ON MULTI-IDEALS AND POLYNOMIAL IDEALS OF BANACH SPACES: A NEW APPROACH TO COHERENCE AND COMPATIBILITY

DANIEL PELLEGRINO AND JOILSON RIBEIRO

Abstract. What is an adequate extension of an operator ideal $\mathcal{I}$ to the polynomial and multilinear settings? This question motivated the appearance of the concepts of coherent sequences of polynomial ideals and compatibility of a polynomial ideal with an operator ideal, introduced by D. Carando *el al.* We propose a different approach by considering pairs $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^{\infty}$, where $(\mathcal{U}_k)_{k=1}^{\infty}$ is a polynomial ideal and $(\mathcal{M}_k)_{k=1}^{\infty}$ is a multi-ideal, instead of considering just polynomial ideals. Our approach ends a discomfort caused by the previous theory: for real scalars the canonical sequence $(\mathcal{P}_k)_{k=1}^{\infty}$ of continuous k-homogeneous polynomials is not coherent according to the definition of Carando *et al.*

We apply these new notions to test the factorization method and different classes that generalise the concept of absolutely summing operator.

1. Introduction and background

The origin of the theory of operator ideals goes back to 1941, with J.W. Calkin [18] and subsequent works of H. Weyl [68] and A. Grothendieck [35] but only in the 70's, with the work of A. Pietsch [62], the basic concepts were organised and presented (see also [27, 39]). For applications of the subject in different areas of mathematics we refer the reader to the recent paper [27] and for more historical details we refer to [64]. The extension to multilinear mappings, with the concept of multi-ideals, is also due to Pietsch [63].

The investigation of special sets of homogeneous polynomials and multilinear mappings between Banach spaces is historically motivated by two different directions: infinite-dimensional holomorphy ([30, 48]) or the theory of operator ideals, polynomial ideals and multi-ideals ([62, 63]). The holomorphic approach, of course, is mostly focused on polynomials rather than on multilinear operators ([19, 20, 49]). In this paper we will be mainly motivated by the theory of operator ideals and for this reason we are also interested in multi-ideals.

Several common multi-ideals and polynomial ideals are usually associated to some operator ideal; however, the extension of an operator ideal to polynomials and multilinear mappings is not always a simple task. For example, the ideal of absolutely summing operators has, at least, eight possible extensions to higher degrees (see, for example, [13, 17, 29, 46, 47, 53, 59] and references therein).

The main goal of this paper is to introduce clear general criteria to decide how multilinear and polynomial generalisations of a given operator ideal must behave in order to be considered "adequate". Our criteria are defined simultaneously to pairs of ideals of polynomials and multi-ideals and, to the best of our knowledge, this is the first attempt in this direction in the

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literature. The arguments used throughout this paper are fairly clear and simple in nature but we do believe that they can shed some light to future works on ideals of operators, polynomials and multilinear mappings.

From now on the letters $E, E_1, ..., E_n, F, G, H$ will represent Banach spaces over the same scalar-field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}_1$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components

$$\mathcal{I}(E;F) := \mathcal{L}_1(E;F) \cap \mathcal{I}$$

satisfy:

(Oa) $\mathcal{I}(E;F)$ is a linear subspace of $\mathcal{L}_1(E;F)$ which contains the finite rank operators.

(Ob) If $u \in \mathcal{I}(E;F)$, $v \in \mathcal{L}_1(G;E)$ and $w \in \mathcal{L}_1(F;H)$, then $w \circ u \circ v \in \mathcal{I}(G;H)$.

The operator ideal is a normed operator ideal if there is a function $\|\cdot\|_{\mathcal{I}} \colon \mathcal{I} \longrightarrow [0,\infty)$ satisfying

(O1) $\|\cdot\|_{\mathcal{I}}$ restricted to $\mathcal{I}(E;F)$ is a norm, for all Banach spaces E, F.

(O2) $\|P_1 \colon \mathbb{K} \longrightarrow \mathbb{K} : P_1(\lambda) = \lambda\|_{\mathcal{I}} = 1$.

(O3) If $u \in \mathcal{I}(E;F)$, $v \in \mathcal{L}_1(G;E)$ and $w \in \mathcal{L}_1(F;H)$, then

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\|\|u\|_{\mathcal{I}}\|v\|.$$

When $\mathcal{I}(E;F)$ with the norm above is always complete, $\mathcal{I}$ is called a Banach operator ideal.

For each positive integer n, let $\mathcal{L}_n$ denote the class of all continuous n-linear operators between Banach spaces. An ideal of multilinear mappings (or multi-ideal) $\mathcal{M}$ is a subclass of the class $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}_n$ of all continuous multilinear operators between Banach spaces such that for a positive integer n, Banach spaces $E_1, \ldots, E_n$ and F, the components

$$\mathcal{M}_n(E_1, \ldots, E_n; F) := \mathcal{L}_n(E_1, \ldots, E_n; F) \cap \mathcal{M}$$

satisfy:

(Ma) $\mathcal{M}_n(E_1, \ldots, E_n; F)$ is a linear subspace of $\mathcal{L}_n(E_1, \ldots, E_n; F)$ which contains the n-linear mappings of finite type.

(Mb) If $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$, $u_j \in \mathcal{L}_1(G_j; E_j)$ for $j = 1, \ldots, n$ and $v \in \mathcal{L}_1(F;H)$, then

$$v \circ T \circ (u_1, \ldots, u_n) \in \mathcal{M}_n(G_1, \ldots, G_n; H).$$

Moreover, $\mathcal{M}$ is a (quasi-) normed multi-ideal if there is a function $\|\cdot\|_{\mathcal{M}} \colon \mathcal{M} \longrightarrow [0,\infty)$ satisfying

(M1) $\|\cdot\|_{\mathcal{M}}$ restricted to $\mathcal{M}_n(E_1, \ldots, E_n; F)$ is a (quasi-) norm, for all Banach spaces $E_1, \ldots, E_n$ and F.

(M2) $\|T_n \colon \mathbb{K}^n \longrightarrow \mathbb{K} : T_n(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n\|_{\mathcal{M}} = 1$ for all n,

(M3) If $T \in \mathcal{M}_n(E_1, \ldots, E_n; F)$, $u_j \in \mathcal{L}_1(G_j; E_j)$ for $j = 1, \ldots, n$ and $v \in \mathcal{L}_1(F;H)$, then

$$\|v \circ T \circ (u_1, \ldots, u_n)\|_{\mathcal{M}} \leq \|v\|\|T\|_{\mathcal{M}}\|u_1\| \cdots \|u_n\|.$$

When all the components $\mathcal{M}_n(E_1,\ldots,E_n;F)$ are complete under this (quasi-) norm, $\mathcal{M}$ is called a (quasi-) Banach multi-ideal. For a fixed multi-ideal $\mathcal{M}$ and a positive integer n, the class

$$\mathcal{M}_n := \cup_{E_1,...,E_n,F}\mathcal{M}_n\left(E_1,...,E_n;F\right)$$

is called ideal of n-linear mappings.

Analogously, for each positive integer n, let $\mathcal{P}_n$ denote the class of all continuous n-homogeneous polynomials between Banach spaces. A polynomial ideal $\mathcal{Q}$ is a subclass of the class $\mathcal{P} = \bigcup\limits_{n=1}^{\infty}\mathcal{P}_n$ of all continuous homogeneous polynomials between Banach spaces so that for all $n\in\mathbb{N}$ and all Banach spaces E and F, the components

$$\mathcal{Q}_n\left({}^nE;F\right) := \mathcal{P}_n\left({}^nE;F\right)\cap\mathcal{Q}$$

satisfy:

(Pa) $\mathcal{Q}_n\left({}^nE;F\right)$ is a linear subspace of $\mathcal{P}_n\left({}^nE;F\right)$ which contains the finite-type polynomials.

(Pb) If $u\in\mathcal{L}_1\left(G;E\right)$, $P\in\mathcal{Q}_n\left({}^nE;F\right)$ and $w\in\mathcal{L}_1\left(F;H\right)$, then

$$w\circ P\circ u\in\mathcal{Q}_n\left({}^nG;H\right).$$

If there exists a map $\left\|\cdot\right\|_{\mathcal{Q}}:\mathcal{Q}\to[0,\infty[$ satisfying

(P1) $\left\|\cdot\right\|_{\mathcal{Q}}$ restricted to $\mathcal{Q}_n({}^nE;F)$ is a (quasi-) norm for all Banach spaces E and F and all n;

(P2) $\left\|P_n:\mathbb{K}\to\mathbb{K};\ P_n\left(\lambda\right)=\lambda^n\right\|_{\mathcal{Q}}=1$ for all n;

(P3) If $u\in\mathcal{L}_1(G;E)$, $P\in\mathcal{Q}_n({}^nE;F)$ and $w\in\mathcal{L}_1(F;H)$, then

$$\left\|w\circ P\circ u\right\|_{\mathcal{Q}}\leq\left\|w\right\|\left\|P\right\|_{\mathcal{Q}}\left\|u\right\|^n,$$

$\mathcal{Q}$ is called (quasi-) normed polynomial ideal. If all components $\mathcal{Q}_n\left({}^nE;F\right)$ are complete, $\left(\mathcal{Q},\left\|\cdot\right\|_{\mathcal{Q}}\right)$ is called a (quasi-) Banach ideal of polynomials (or (quasi-) Banach polynomial ideal). For a fixed ideal of polynomials $\mathcal{Q}$ and $n\in\mathbb{N}$, the class

$$\mathcal{Q}_n := \cup_{E,F}\mathcal{Q}_n\left({}^nE;F\right)$$

is called ideal of n-homogeneous polynomials.

A relevant question in the multilinear/polynomial setting is: given an operator ideal (for example the class of compact operators, weakly compact operators, absolutely summing operators, strictly singular operators, etc) how to define a multi-ideal and a polynomial ideal related to the linear operator ideal without loosing its essence? How to lift the core of an operator ideal to polynomials and multilinear mappings?

The crucial point is that in general a given operator ideal has several different possible extensions to the setting of multi-ideals and polynomial ideals (see, for example the case of absolutely summing operators [17, 56, 59] and almost summing operators [9, 50, 52]). In order to have a method of evaluating what kind of multilinear/polynomial extensions of a given operator ideal is more adequate and less artificial, some efforts have been done in the last years, but these efforts were not addressed simultaneously to polynomials and multilinear mappings.

In this paper we intend to contribute in the direction of identifying the precise properties that polynomial and multi-ideals must fulfill (simultaneously) in order to keep the essence of a given operator ideal. The paper is organised as follows:

In Section 2 we recall the concepts already created for this task (ideals closed for scalar multiplication, ideals closed under differentiation, holomorphy types, coherent ideals, compatible ideals).

In Section 3 we present our new approach to coherence and compatibility of pairs and in the subsequent sections we test our approaches to the factorisation method and for pairs of ideals that generalise the ideal of absolutely summing operators. In the last section we sketch a stronger notion of coherence and compatibility.

2. Coherent polynomial ideals, global holomorphy types and related concepts

In this section we recall old and recent concepts that, in some sense, evaluate how good a multilinear/polynomial extension of an operator ideal is. The concepts of ideals of polynomials closed under differentiation and closed for scalar multiplication were introduced in [11] (see also [10] for related notions) as an attempt of identifying crucial properties that polynomial ideals must verify in order to maintain some harmony between the different levels of homogeneity.

From now on E^* denotes the topological dual of E and we use the following notation:

- If $P \in \mathcal{P}_n\left({}^nE;F\right)$, then $\overset{\vee}{P}$ denotes the unique symmetric n-linear mapping associated to P.
- If $P \in \mathcal{P}_n\left({}^nE;F\right)$, then $P_{a^k} \in \mathcal{P}_{n-k}\left({}^{n-k}E;F\right)$ is defined by
$$P_{a^k}(x) := \overset{\vee}{P}(a,...,a,x,...,x).$$
- If $T \in \mathcal{L}_n\left(E_1,\ldots,E_n;F\right)$, then $T_{a_1,\ldots,a_{k-1}a_{k+1}\ldots a_n} \in \mathcal{L}_1\left(E_k;F\right)$ denotes the mapping
$$T_{a_1,\ldots,a_{k-1}a_{k+1}\ldots a_n}(x_k) := T(a_1,...,a_{k-1},x_k,a_{k+1},...,a_n).$$
- If $T \in \mathcal{L}_n\left(E_1,\ldots,E_n;F\right)$, then $T_{a_j} \in \mathcal{L}_{n-1}\left(E_1,...,E_{j-1},E_{j+1},...,E_n;F\right)$ is given by
$$T_{a_j}(x_1,...,x_{j-1},x_{j+1},...,x_n) := T(x_1,...,x_{j-1},a_j,x_{j+1},...,x_n).$$

Definition 2.1 (csm and cud polynomial ideals [11])**.** *Let $\mathcal{Q}$ be a polynomial ideal, $n \in \mathbb{N}$, E and F Banach spaces.*

(**i**) *$\mathcal{Q}$ is closed under differentiation (cud) for n, E and F if $\hat{d}P\left(a\right) \in \mathcal{Q}_1\left(E;F\right)$ for all $a \in E$ and $P \in \mathcal{Q}_n\left({}^nE;F\right)$, where $\hat{d}P\left(a\right)$ is the derivative of P at a.*

(**ii**) *$\mathcal{Q}$ is closed for scalar multiplication (csm) for n, E and F if $\varphi P \in \mathcal{Q}_{n+1}\left({}^{n+1}E;F\right)$ for all $\varphi \in E^*$ and $P \in \mathcal{Q}_n\left({}^nE;F\right)$.*

When (**i**) *and/or* (**ii**) *holds true for all n, E and F we say that $\mathcal{Q}$ is cud and/or csm.*

The same concept can be translated, *mutatis mutandis*, to multilinear mappings.

Recently, in [19] (for related papers see also [20, 21]), the interesting notions of compatible polynomial ideals and coherent ideals were presented with the same aim of filtering good polynomial extensions of given operator ideals. This approach (with its nice self-explanatory terminology) offers, with the notion of compatibility, a clear proposal of classifying when a polynomial ideal has the spirit of a given operator ideal (our notation essentially follows [19]):

Definition 2.2 (Compatible polynomial ideals [19])**.** *Let $\mathcal{U}$ be a normed ideal of linear operators. The normed ideal of n-homogeneous polynomials $\mathcal{U}_n$ is compatible with $\mathcal{U}$ if there exist positive constants α_1 and α_2 such that for all Banach spaces E and F, the following conditions hold:*

(**i**) *For each $P \in \mathcal{U}_n\left({}^nE;F\right)$ and $a \in E$, $P_{a^{n-1}}$ belongs to $\mathcal{U}\left(E;F\right)$ and*

$$\left\|P_{a^{n-1}}\right\|_{\mathcal{U}} \leq \alpha_1 \left\|P\right\|_{\mathcal{U}_n} \left\|a\right\|^{n-1}$$

(**ii**) *For each $u \in \mathcal{U}\left(E;F\right)$ and $\gamma \in E^*$, $\gamma^{n-1}u$ belongs to $\mathcal{U}_n\left({}^nE;F\right)$ and*

$$\left\|\gamma^{n-1}u\right\|_{\mathcal{U}_n} \leq \alpha_2 \left\|\gamma\right\|^{n-1} \left\|u\right\|_{\mathcal{U}}$$

Remark 2.3. *Sometimes, for the sake of simplicity, we will say that the sequence $(\mathcal{U}_k)_{k=1}^N$ is compatible with $\mathcal{U}$ (this will mean that $\mathcal{U}_k$ is compatible with $\mathcal{U}$ for all $k = 1, ..., N$).*

Definition 2.4 (Coherent polynomial ideals [19])**.** *Consider a sequence $(\mathcal{U}_k)_{k=1}^N$, where for each k, $\mathcal{U}_k$ is an ideal of k-homogeneous polynomials and N is eventually infinity. The sequence $(\mathcal{U}_k)_{k=1}^N$ is a coherent sequence of polynomial ideals if there exist positive constants β_1 and β_2 such that for all Banach spaces the following conditions hold for $k = 1, ..., N-1$:*

(**i**) *For each $P \in \mathcal{U}_{k+1}\left({}^{k+1}E;F\right)$ and $a \in E$, P_a belongs to $\mathcal{U}_k\left({}^kE;F\right)$ and*

$$\left\|P_a\right\|_{\mathcal{U}_k} \leq \beta_1 \left\|P\right\|_{\mathcal{U}_{k+1}} \left\|a\right\|.$$

(**ii**) *For each $P \in \mathcal{U}_k\left({}^kE;F\right)$ and $\gamma \in E^*$, γP belongs to $\mathcal{U}_{k+1}\left({}^{k+1}E;F\right)$ and*

$$\left\|\gamma P\right\|_{\mathcal{U}_{k+1}} \leq \beta_2 \left\|\gamma\right\| \left\|P\right\|_{\mathcal{U}_k}$$

The philosophy of the concepts above is that given positive integers n_1 and n_2, the respective levels of n_1-linearity and n_2-linearity of a given multi-ideal (or polynomial ideal) must show some relevant inter-connection and also keep the spirit of the original level ($n = 1$).

There is no doubt that the concepts of compatible polynomial ideals and coherent ideals have added important contribution to the theory of polynomial ideals. However, an operator ideal $\mathcal{I}$ can be always extended to the multilinear and polynomial settings (at least in an abstract sense; for details see [8]); so, there is no apparent reason to consider the concepts of compatibility and coherence just for polynomials (or just for multilinear mappings separately). Our proposal offers significant variations of these notions (as it will be clear in the next paragraph) by considering pairs $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^\infty$, where $(\mathcal{U}_k)_{k=1}^\infty$ is a polynomial ideal and $(\mathcal{M}_k)_{k=1}^\infty$ is a multi-ideal. So, this new approach deals simultaneously with polynomials and multilinear operators and, of course, asks for some harmony between $(\mathcal{U}_k)_{k=1}^\infty$ and $(\mathcal{M}_k)_{k=1}^\infty$.

It is very important to recall that, for the case of real scalars, the canonical sequence $(\mathcal{P}_k)_{k=1}^\infty$, composed by the ideals of continuous k-homogeneous polynomials with the sup norm, is not coherent according to Definition 2.4 (this remark appears in [19] and is based on estimates for the norms of certain special homogeneous polynomials used in [10, Proposition 8.5]). This result seems to be uncomfortable since the canonical sequence $(\mathcal{P}_k)_{k=1}^\infty$ should be a prototype of the essence of coherence. Using our forthcoming approach to the notion of coherence, the pair $(\mathcal{P}_k, \mathcal{L}_k)_{k=1}^\infty$ (composed by the ideals of continuous n-homogeneous polynomials and continuous n-linear operators, with the sup norm) will be coherent and compatible with the ideal of continuous linear operators.

It is worth mentioning that all these concepts have some connection with the concept of Property (B) defined in [10] and also with the idea of holomorphy types, due to L. Nachbin (and with a similar notion of global holomorphy types, which appears in [10]). In [19] the reader can find some useful comparisons between the concepts above. As mentioned in [19], coherent sequences are always global holomorphy types. It is interesting to note that in [19, Example 1.15] the exact example that fails to be coherent/compatible with the ideal of absolutely summing operators also fails to be a global holomorphy type. For recent striking applications of the theory of holomorphy types we refer to [5].

3. Coherence and compatibility: a new approach

From now on $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$ is a sequence, where each $\mathcal{U}_k$ is a (quasi-) normed ideal of k-homogeneous polynomials and each $\mathcal{M}_k$ is a (quasi-) normed ideal of k-linear mappings. The parameter N can be eventually infinity. Motivated by the argument that the notions of compatibility and coherence should be defined simultaneously for polynomials and multilinear mappings, we propose the following approach:

Definition 3.1 (Compatible pair of ideals)**.** *Let $\mathcal{U}$ be a normed operator ideal and $N \in (\mathbb{N} \setminus \{1\}) \cup \{\infty\}$. A sequence $(\mathcal{U}_n, \mathcal{M}_n)_{n=1}^N$ with $\mathcal{U}_1 = \mathcal{M}_1 = \mathcal{U}$ is compatible with $\mathcal{U}$ if there exist positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that for all Banach spaces E and F, the following conditions hold for all $n \in \{2, ..., N\}$:*

(CP 1) If $k \in \{1, \dots, n\}$, $T \in \mathcal{M}_n(E_1, ..., E_n; F)$ and $a_j \in E_j$ for all $j \in \{1, ..., n\} \setminus \{k\}$, then

$$T_{a_1,\dots,a_{k-1},a_{k+1},\dots,a_n} \in \mathcal{U}(E_k; F)$$

and

$$\left\| T_{a_1,\dots,a_{k-1},a_{k+1},\dots,a_n} \right\|_{\mathcal{U}} \leq \alpha_1 \|T\|_{\mathcal{M}_n} \|a_1\| \dots \|a_{k-1}\| \|a_{k+1}\| \dots \|a_n\| .$$

(CP 2) If $P \in \mathcal{U}_n({}^nE; F)$ and $a \in E$, then $P_{a^{n-1}} \in \mathcal{U}(E; F)$ and

$$\|P_{a^{n-1}}\|_{\mathcal{U}} \leq \alpha_2 \left\| \overset{\vee}{P} \right\|_{\mathcal{M}_n} \|a\|^{n-1} .$$

(CP 3) If $u \in \mathcal{U}(E_n; F)$, $\gamma_j \in E_j^*$ for all $j = 1, ..., n-1$, then

$$\gamma_1 \cdots \gamma_{n-1} u \in \mathcal{M}_n(E_1, .., E_n; F)$$

and

$$\|\gamma_1 \cdots \gamma_{n-1} u\|_{\mathcal{M}_n} \leq \alpha_3 \|\gamma_1\| ... \|\gamma_{n-1}\| \|u\|_{\mathcal{U}} .$$

(CP 4) If $u \in \mathcal{U}(E; F)$, $\gamma \in E^*$, then $\gamma^{n-1} u \in \mathcal{U}_n({}^nE; F)$ and

$$\left\|\gamma^{n-1} u\right\|_{\mathcal{U}_n} \leq \alpha_4 \|\gamma\|^{n-1} \|u\|_{\mathcal{U}} .$$

(CP 5) P belongs to $\mathcal{U}_n({}^nE; F)$ if, and only if, $\overset{\vee}{P}$ belongs to $\mathcal{M}_n({}^nE; F)$.

Definition 3.2 (Coherent pair of ideals)**.** *Let $\mathcal{U}$ be a normed operator ideal and $N \in \mathbb{N} \cup \{\infty\}$. A sequence $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$, with $\mathcal{U}_1 = \mathcal{M}_1 = \mathcal{U}$, is coherent if there exist positive constants $\beta_1, \beta_2, \beta_3, \beta_4$ such that for all Banach spaces E and F the following conditions hold for $k = 1, ..., N-1$:*

(CH 1) If $T \in \mathcal{M}_{k+1}(E_1, ..., E_{k+1}; F)$ *and* $a_j \in E_j$ *for* $j = 1, \ldots, k+1$, *then*
$$T_{a_j} \in \mathcal{M}_k(E_1, \ldots, E_{j-1}, E_{j+1}, \ldots, E_{k+1}; F)$$
and
$$\left\|T_{a_j}\right\|_{\mathcal{M}_k} \leq \beta_1 \|T\|_{\mathcal{M}_{k+1}} \|a_j\|.$$
(CH 2) If $P \in \mathcal{U}_{k+1}\left({}^{k+1}E; F\right)$, $a \in E$, *then* P_a *belongs to* $\mathcal{U}_k\left({}^kE; F\right)$ *and*
$$\|P_a\|_{\mathcal{U}_k} \leq \beta_2 \left\|\overset{\vee}{P}\right\|_{\mathcal{M}_{k+1}} \|a\|.$$
(CH 3) If $T \in \mathcal{M}_k(E_1, ..., E_k; F)$, $\gamma \in E_{k+1}^*$, *then*
$$\gamma T \in \mathcal{M}_{k+1}(E_1, ..., E_{k+1}; F) \text{ and } \|\gamma T\|_{\mathcal{M}_{k+1}} \leq \beta_3 \|\gamma\| \|T\|_{\mathcal{M}_k}.$$
(CH 4)If $P \in \mathcal{U}_k\left({}^kE; F\right)$, $\gamma \in E^*$, *then*
$$\gamma P \in \mathcal{U}_{k+1}\left({}^{k+1}E; F\right) \text{ and } \|\gamma P\|_{\mathcal{U}_{k+1}} \leq \beta_4 \|\gamma\| \|P\|_{\mathcal{U}_k}.$$
(CH 5) For all $k = 1, ..., N$, P *belongs to* $\mathcal{U}_k\left({}^kE; F\right)$ *if, and only if,* $\overset{\vee}{P}$ *belongs to* $\mathcal{M}_k\left({}^kE; F\right)$.

Remark 3.3. *Note that Definition 3.1 is quite different from the concept of compatibility from Carando et al. For example, our approach asks for universal constants* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *(that not depend on* n*). It is also worth mentioning that a coherent sequence* $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$ *is not necessarily compatible with* $\mathcal{U}_1$*. If* $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$ *then the coherence of a sequence* $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$ *easily implies in the compatibility with* $\mathcal{U}_1$*. In the Example 8.2 we also show that a sequence* $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$ *which is compatible with* $\mathcal{U}_1$ *may fail to be coherent.*

A weaker concept of coherence of pairs, where the constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *may depend on* k *may be also interesting.*

As we have mentioned before, for the real case the canonical sequence $(\mathcal{P}_k)_{k=1}^\infty$ is not coherent according to the original definition of Carando, Dimant and Muro. The following straightforward proposition shows that the situation is different according to the new approach:

Proposition 3.4. *The pair* $(\mathcal{P}_k, \mathcal{L}_k)_{k=1}^\infty$ *(composed by the ideals of continuous* n*-homogeneous polynomials and continuous* n*-linear operators, with the* sup *norm) is coherent and compatible with the ideal of continuous linear operators.*

4. The Factorisation Method

The factorisation method is an important abstract way of extending an operator ideal to polynomials and multilinear mappings. In this section we show that the sequence of pairs obtained by this method is coherent and compatible with the original ideal.

For an operator ideal $\mathcal{I}$, an n-linear mapping $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is of type $\mathcal{L}({}^n\mathcal{I})$ if there are Banach spaces $G_1, \ldots, G_n$, linear operators $u_j \in \mathcal{I}(E_j, G_j)$, $j = 1, \ldots, n$ and $B \in \mathcal{L}(G_1, \ldots, G_n; F)$ such that

$$A = B \circ (u_1, \ldots, u_n). \tag{4.1}$$

In this case we write
$$A \in \mathcal{L}({}^n\mathcal{I})(E_1, \ldots, E_n; F),$$

and define

$$\|A\|_{\mathcal{L}(^n\mathcal{I})} = \inf \|B\| \, \|u_1\|_{\mathcal{I}}, \ldots, \|u_n\|_{\mathcal{I}},$$

where the infimum is taken over all possible factorisations (4.1).

The following definition will be useful:

Definition 4.1. *If $\mathcal{M} = (\mathcal{M}_n)_{n=1}^{\infty}$ is a (quasi-) normed multi-ideal, $(\mathcal{P}_{\mathcal{M}}^n)_{n=1}^{\infty}$ is a sequence such that $P \in \mathcal{P}_{\mathcal{M}}^n(^nE;F)$ if and only if $\overset{\vee}{P} \in \mathcal{M}_n(^nE;F)$. Moreover*

$$\|P\|_{\mathcal{P}_{\mathcal{M}}^n} := \left\|\overset{\vee}{P}\right\|_{\mathcal{M}_n}.$$

Proposition 4.2. *(see [10, page 46]) If $\mathcal{M} = (\mathcal{M}_n)_{n=1}^{\infty}$ is a (quasi-) Banach multi-ideal then $\mathcal{P}_{\mathcal{M}} := (\mathcal{P}_{\mathcal{M}}^k)_{k=1}^{\infty}$ is a (quasi-) Banach polynomial ideal.*

For all n, $\mathcal{L}(^n\mathcal{I})$ is a complete $(1/n)$-normed ideal of n-linear mappings; hence $(\mathcal{L}(^n\mathcal{I}))_{n=1}^{\infty}$ is a quasi-Banach multi-ideal and $\left(\mathcal{P}_{\mathcal{L}(^n\mathcal{I})}^n\right)_{n=1}^{\infty}$, constructed as in Definition 4.1, is a quasi-Banach polynomial ideal.

The proof that the sequence $\left(\left(\mathcal{P}_{\mathcal{L}(^n\mathcal{I})}^n, \|.\|_{\mathcal{P}_{\mathcal{L}(^n\mathcal{I})}^n}\right), \left(\mathcal{L}(^n\mathcal{I}), \|.\|_{\mathcal{L}(^n\mathcal{I})}\right)\right)_{n=1}^{\infty}$ is coherent and compatible with the ideal $\mathcal{I}$ will be an immediate consequence of the next results.

Proposition 4.3. *If $P \in \mathcal{P}_{\mathcal{L}(^{n+1}\mathcal{I})}^{n+1}\left(^{n+1}E;F\right)$ and $a \in E$, then P_a belongs to $\mathcal{P}_{\mathcal{L}(^n\mathcal{I})}^n(^nE;F)$ and*

$$\|P_a\|_{\mathcal{P}_{\mathcal{L}(^n\mathcal{I})}^n} \leq \left\|\overset{\vee}{P}\right\|_{\mathcal{L}(^{n+1}\mathcal{I})} \|a\|.$$

Proof. There exist Banach spaces $G_1, ..., G_{n+1}$, linear operators $u_j \in \mathcal{I}_j(E;G_j)$, $j = 1, \ldots, n+1$, and a multilinear mapping $B \in \mathcal{L}(G_1, ..., G_{n+1}; F)$ such that

$$\overset{\vee}{P}(x_1, \ldots, x_{n+1}) = B(u_1(x_1), \ldots, u_{n+1}(x_{n+1})). \tag{4.2}$$

So

$$\begin{aligned}(P_a)^{\vee}(x_1, ..., x_n) = \overset{\vee}{P}(x_1, \ldots, x_n, a) &= B(u_1(x_1), \ldots, u_n(x_n), u_{n+1}(a)) \\ &= B_{u_{n+1}(a)}(u_1(x_1), \ldots, u_n(x_n)).\end{aligned}$$

Hence, since $\|u_{n+1}\| \leq \|u_{n+1}\|_{\mathcal{I}}$, we have

$$\|P_a\|_{\mathcal{P}_{\mathcal{L}(^n\mathcal{I})}^n} = \left\|(P_a)^{\vee}\right\|_{\mathcal{L}(^n\mathcal{I})} \leq \left\|B_{u_{n+1}(a)}\right\| \prod_{j=1}^{n} \|u_j\|_{\mathcal{I}} \leq \|a\| \, \|B\| \prod_{j=1}^{n+1} \|u_j\|_{\mathcal{I}}.$$

for all representation (4.2) and the proof is completed when considering the infimum over all such representations. ☐

The next result is inspired in [19, Proposition 3.1]:

Proposition 4.4. *If $P \in \mathcal{P}_{\mathcal{L}(^n\mathcal{I})}^n(^nE;F)$ and $\gamma \in E^*$, then $\gamma P \in \mathcal{P}_{\mathcal{L}(^{n+1}\mathcal{I})}\left(^{n+1}E;F\right)$ and*

$$\|\gamma P\|_{\mathcal{P}_{\mathcal{L}(^{n+1}\mathcal{I})}^{n+1}} \leq \|\gamma\| \, \|P\|_{\mathcal{L}(^n\mathcal{I})}.$$

Proof. We can suppose $\|\gamma\| = 1$. There exist Banach spaces $G_1, \ldots, G_n$, a multilinear mapping $B \in \mathcal{L}(G_1, \ldots, G_n; F)$, and linear operators $u_j \in \mathcal{I}(E; G_j)$, $j = 1, \ldots, n$ such that

$$\overset{\vee}{P} = B \circ (u_1, \ldots, u_n).$$

Now, consider the mapping $\tilde{B} \in \mathcal{L}(G_1, \ldots, G_n, \mathbb{K}; F)$ defined by

$$\tilde{B}(y_1, \ldots, y_n, \gamma(x)) = \gamma(x) B(y_1, \ldots, y_n).$$

Observe that $\tilde{B}$ is well-defined, and that

$$\begin{aligned} \tilde{B}(u_1(x_1), \ldots, u_n(x_n), \gamma(x_{n+1})) &= \gamma(x_{n+1}) B(u_1(x_1), \ldots, u(x_n)) \\ &= \gamma(x_{n+1}) \overset{\vee}{P}(x_1, \ldots, x_n). \end{aligned}$$

Therefore,

$$\gamma \overset{\vee}{P} \in \mathcal{L}\left({}^{n+1}\mathcal{I}\right)\left({}^{n+1}E; F\right).$$

Since $\left\|\tilde{B}\right\| = \|B\|$ and

$$(\gamma P)^{\vee}(x_1, ..., x_{n+1}) = \frac{\gamma(x_1)\overset{\vee}{P}(x_2, ..., x_{n+1}) + ... + \gamma(x_{n+1})\overset{\vee}{P}(x_1, ..., x_n)}{n+1},$$

we have

$$\|\gamma P\|_{\mathcal{P}^{n+1}_{\mathcal{L}(^{n+1}\mathcal{I})}} = \left\|(\gamma P)^{\vee}\right\|_{\mathcal{L}(^{n+1}\mathcal{I})} \leq \frac{1}{n+1}\left((n+1)\left\|\gamma \overset{\vee}{P}\right\|_{\mathcal{L}(^{n+1}\mathcal{I})}\right) \leq \|B\| \prod_{j=1}^{n} \|u_j\|_{\mathcal{I}}$$

and the proof is completed. □

Proposition 4.5. *Let $k \in \{1, \ldots, n+1\}$. If $T \in \mathcal{L}\left({}^{n+1}\mathcal{I}\right)(E_1, ..., E_{n+1}; F)$ and $a_k \in E_k$, then*

$$T_{a_k} \in \mathcal{L}({}^{n}\mathcal{I})(E_1, \ldots, E_{k-1}, E_{k+1}, \ldots, E_{n+1}; F)$$

and

$$\|T_{a_k}\|_{\mathcal{L}(^{n}\mathcal{I})} \leq \|T\|_{\mathcal{L}(^{n+1}\mathcal{I})} \|a_k\|.$$

Proof. There are Banach spaces $G_1, \ldots, G_{n+1}$, linear operators $u_j \in \mathcal{I}(E_j; G_j)$, $j = 1, \ldots, n+1$, and $B \in \mathcal{L}(G_1, \ldots, G_{n+1}; F)$ such that

$$T(x_1, \ldots, x_{n+1}) = B(u_1(x_1), \ldots, u_{n+1}(x_{n+1})).$$

Hence,

$$\begin{aligned} &T_{a_k}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots x_{n+1}) \\ &= B_{u_k(a_k)}(u_1(x_1), \ldots, u_{k-1}(x_{k-1}), u_{k+1}(x_{k+1}), \ldots, u_{n+1}(x_{n+1})) \end{aligned}$$

and the proof follows the lines of the proof of Proposition 4.4. □

The proof of the next proposition is essentially the same of Proposition 4.4, and we omit:

Proposition 4.6. *If $T \in \mathcal{L}({}^{n}\mathcal{I})(E_1, \ldots, E_n; F)$ and $\gamma \in E_{n+1}^{*}$, then*

$$\gamma T \in \mathcal{L}\left({}^{n+1}\mathcal{I}\right)(E_1, ..., E_{n+1}; F) \text{ and } \|\gamma T\|_{\mathcal{L}(^{n+1}\mathcal{I})} \leq \|\gamma\| \|T\|_{\mathcal{L}(^{n}\mathcal{I})}.$$

From the previous results and Remark 3.3 we have:

Theorem 4.7. *The sequence* $\left(\left(\mathcal{P}^n_{\mathcal{L}(^n\mathcal{I})}, \|.\|_{\mathcal{P}^n_{\mathcal{L}(^n\mathcal{I})}}\right), \left(\mathcal{L}\left(^n\mathcal{I}\right), \|.\|_{\mathcal{L}(^n\mathcal{I})}\right)\right)_{n=1}^{\infty}$ *is coherent and compatible with the ideal* $\mathcal{I}$.

5. Nonlinear variants of absolutely summing operators: a brief summary

The class of absolutely p-summing linear operators is one of the most successful examples of operator ideals, having its special space in several textbooks related to Banach Space Theory (we refer the reader to [1, 26, 32, 44, 67, 69]). Its roots go back to the 1950s, when Grothendieck still worked in Functional Analysis (see the famous Résumé [34] and also [35, 36, 37, 38] for other papers of Grothendieck in Functional Analysis). The classical papers of Lindenstrauss–Pełczyński [43] and Pietsch [61] were also fundamental for the development of the theory.

As we have mentioned before, there are various possible multilinear and polynomial approaches to the notion of absolutely summing operators; for this reason we think that this is a nice context to test the notions of coherence and compatibility of pairs previously defined.

For $1 \leq p < \infty$, we denote by $\ell_p^w(E)$ the space composed by the sequences $(x_j)_{j=1}^{\infty}$ in E so that $(\varphi(x_j))_{j=1}^{\infty} \in \ell_p$ for all continuous linear functionals $\varphi : E \to \mathbb{K}$. The space $\ell_p^w(E)$ is a Banach space when endowed with the norm $\|\cdot\|_{w,p}$ given by

$$\left\|(x_j)_{j=1}^{\infty}\right\|_{w,p} := \sup_{\varphi \in B_{E^*}} \left\|(\varphi(x_j))_{j=1}^{\infty}\right\|_p .$$

The subspace of $\ell_p^w(E)$ of all sequences $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$ such that $\lim_{m\to\infty} \left\|(x_j)_{j=m}^{\infty}\right\|_{w,p} = 0$ is denoted by $\ell_p^u(E)$. If $1 \leq q \leq p < \infty$ a continuous linear operator $u : E \to F$ is absolutely (p,q)-summing if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{\infty} \|u(x_j)\|^p\right)^{1/p} \leq C \left\|(x_j)_{j=1}^{\infty}\right\|_{w,q}$$

for all $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$. We denote the set of all absolutely (p,q)-summing operators by $\Pi_{p,q}$ (Π_p if $p = q$) and the space of all absolutely (p,q)-summing operators from E to F by $\Pi_{p,q}(E,F)$. The infimum of all C that satisfy the inequality above defines a norm on $\Pi_{p,q}(E,F)$, denoted by $\|.\|_{as(p,q)}$ (or $\|.\|_{as,p}$ if $p = q$) and $\left(\Pi_{p,q}, \|.\|_{as(p,q)}\right)$ is a Banach operator ideal. The notion of absolutely summing operators is due to Pietsch [61]. For a complete panorama on the theory of absolutely summing operators we refer the reader to the classical monograph [26] and classical papers [4, 25, 31, 43, 61]; for recent developments we refer the reader to [15, 40, 41, 42, 66] and references therein.

The adequate extension of the linear theory of absolutely summing operators to the multilinear setting is a complicated matter; there are different approaches and different lines of investigation.

Historically, in some sense, the multilinear theory of absolutely summing mappings seems to have its starting point in 1930, with Littlewood's 4/3 Theorem [45] and, one year later, with the Bohnenblust–Hille Theorem [6]. The Bohnenblust–Hille Theorem was overlooked for a long time and only in the the 80's the interest in the multilinear theory related to absolutely summing operators was recovered, motivated by A. Pietsch's work [63]. In the present paper we deal

with some of the most usual polynomial and multilinear extensions of Π_p (strongly summing multilinear operators, multiple summing multilinear operators, absolutely summing multilinear operators, dominated multilinear operators, everywhere absolutely summing multilinear operators and their polynomial versions). For more details concerning the nonlinear theory of absolutely summing operators and recent developments and applications we refer the reader to [10, 14, 22, 23, 29, 47, 51, 60] and references therein.

A polynomial $P \in \mathcal{P}_n({}^nE; F)$ is $(p;q)$-summing at $a \in E$ if $(P(a+x_j) - P(a))_{j=1}^{\infty} \in \ell_p(F)$ for all $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$. If $1 \leq q \leq p < \infty$, the space composed by the n-homogeneous polynomials from E to F that are $(p;q)$-summing at every point is denoted by $\mathcal{P}_{as(p;q)}^{n,ev}({}^nE; F)$. The polynomials in $\mathcal{P}_{as(p;q)}^{n,ev}$ are called everywhere absolutely summing.

M.C. Matos [47] defined a norm on the space $\mathcal{P}_{as(p;q)}^{n,ev}({}^nE; F)$ of everywhere $(p;q)$-summing polynomials by considering the polynomial $\Psi_{p;q}(P) \colon \ell_q^u(E) \longrightarrow \ell_p(F)$ given by

$$(x_j)_{j=1}^{\infty} \longmapsto (P(x_1), (P(x_1 + x_j) - P(x_1))_{j=2}^{\infty})$$

and showing that the correspondence $P \longrightarrow \|\Psi_{p;q}(P)\|$ defines a norm on $\mathcal{P}_{as(p;q)}^{n,ev}({}^nE; F)$. From now on this norm is denoted by $\|\cdot\|_{ev^{(1)}(p;q)}$. Matos also proved that this norm is complete and that $(\mathcal{P}_{as(p;q)}^{n,ev}, \|\cdot\|_{ev^{(1)}(p;q)})$ is a global holomorphy type. From [3] it is known that

$$\lim_{n\to\infty} \|P_n : \mathbb{K} \to \mathbb{K};\ P_n(\lambda) = \lambda^n\|_{ev^{(1)}(p;q)} = \infty$$

and this estimate will allow us to conclude that $(\mathcal{P}_{as(p;q)}^{n,ev}, \|\cdot\|_{ev^{(1)}(p;q)})_{n=1}^{\infty}$ is "compatible, in the sense of Carando *et al.*" with no operator ideal; here we have used the term "compatible , in the sense of Carando *et al.*" in a more general form, since the sequence $(\mathcal{P}_{as(p;q)}^{n,ev}, \|\cdot\|_{ev^{(1)}(p;q)})_{n=1}^{\infty}$ is not exactly a normed polynomial ideal (since it fails (P2)), but just a global holomorphy type.

Proposition 5.1. *If $p \geq q > 1$ and $n \geq 2$, then $(\mathcal{P}_{as(p;q)}^{n,ev}, \|\cdot\|_{ev^{(1)}(p;q)})$ is "compatible" with no operator ideal.*

Proof. Given $n \in \mathbb{N}$, let $P_n \colon \mathbb{K} \longrightarrow \mathbb{K}$ denote the trivial n-homogeneous polynomial given by $P_n(\lambda) = \lambda^n$. From [3, Proposition 4.4] it is also known that

$$\lim_{n\to\infty} \|P_n\|_{ev^{(1)}(p;q)} = \infty$$

for all $p \geq q > 1$. So, by considering $u = \gamma = P_1$, we conclude that $\gamma^{n-1}u$ belongs to $\mathcal{P}_{as(p;q)}^{n,ev}({}^nE; \mathbb{K})$ and

$$\lim_{n\to\infty} \left\|\gamma^{n-1}u\right\|_{ev^{(1)}(p;q)} = \lim_{n\to\infty} \|P_n\|_{ev^{(1)}(p;q)} = \infty$$

and, on the other hand, for every operator ideal $\mathcal{I}$, we have

$$\|\gamma\|^{n-1} \|u\|_{\mathcal{I}} < \infty.$$

We thus conclude that $(\mathcal{P}_{as(p;q)}^{n,ev}, \|\cdot\|_{ev^{(1)}(p;q)})$ is compatible with no operator ideal. $\square$

In the result above the non-compatibility is a fault of the norm and by considering the similar concept for multilinear mappings, the respective pair of ideals will also fail to be compatible

with respect to our new approach. The situation will be different when considering a new norm in the next section.

From now on we will adopt the classical notation for the spaces of continuous n-homogeneous polynomials and continuous n-linear mappings. More precisely, we will write $\mathcal{P}\left({}^{n}E;F\right)$ and $\mathcal{L}\left(E_1,..,E_n;F\right)$ instead of $\mathcal{P}_n\left({}^{n}E;F\right)$ and $\mathcal{L}_n\left(E_1,..,E_n;F\right)$, respectively. When $E_1=\cdots=E_n$ we will write $\mathcal{L}\left({}^{n}E;F\right)$.

6. Everywhere absolutely summing multilinear operators and polynomials

One of the first multilinear generalisations of the ideal of absolutely summing operators (see [2]) is the following: If $0<p,q<\infty$ and $p\geq nq$ an n-homogeneous polynomial $P\in\mathcal{P}({}^{n}E;F)$ is absolutely $(p;q)$-summing if there exists C such that

$$\left(\sum_{j=1}^{\infty}\|P(x_j)\|^p\right)^{1/p}\leq C\left\|(x_j)_{j=1}^{\infty})\right\|_{w,q}^{n} \tag{6.1}$$

for all $(x_j)_{j=1}^{\infty}\in\ell_q^u(E)$. The infimum of all C for which (6.1) holds defines a complete norm (if $p\geq 1$), defined by $\|.\|_{as(p;q)}$, on $\mathcal{P}_{as(p;q)}^{n}({}^{n}E;F)$. If $p=q$ we write $\mathcal{P}_{as,p}^{n}$ instead of $\mathcal{P}_{as(p;q)}^{n}$.

It is not difficult to show that the definition above is equivalent to saying that $(P(x_j))_{j=1}^{\infty}\in\ell_p(F)$ for all $(x_j)_{j=1}^{\infty}\in\ell_q^u(E)$. This ideal, however, is not closed under differentiation and is not a (global) holomorphy type. Besides, in this case the spirit of the linear ideal is also destroyed by several coincidence theorems, which have no relation with the linear case. For example, using that ℓ_p (for $1\leq p\leq 2$) has cotype 2 it is easy to show that

$$\mathcal{P}_{as,1}^{n}({}^{n}\ell_p;F)=\mathcal{P}({}^{n}\ell_p;F) \tag{6.2}$$

for all $p\in[1,2]$, $n\geq 2$ and all F; these results are far from being true for $n=1$. So it should be expected that $\left(\mathcal{P}_{as,1}^{n},\|.\|_{as,1}\right)$ is not classified as compatible with the ideal Π_1. Similar defects can be found in this ideal for the general case of $\mathcal{P}_{as(p,q)}^{n}$. A similar concept of absolutely summability exists for n-linear operators:

An n-linear operator $T\in\mathcal{L}(E_1,\ldots,E_n;F)$ is absolutely $(p;q)$-summing (with $p\geq nq$) if there exists $C\geq 0$ such that

$$\left(\sum_{j=1}^{\infty}\left\|T(x_j^{(1)},\ldots,x_j^{(n)})\right\|^p\right)^{1/p}\leq C\prod_{k=1}^{n}\left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,q}$$

for all $(x_j^{(k)})_{j=1}^{\infty}\in\ell_q^u(E_k)$, $k=1,\ldots,n$. Moreover, the infimum of all C for which the inequality holds defines a complete norm (if $p\geq 1$), denoted by $\|.\|_{as(p;q)}$, for this class. This definition is equivalent to saying that $\left(T(x_j^{(1)},\ldots,x_j^{(n)})\right)_{j=1}^{\infty}$ belongs to $\ell_p(F)$ for all $(x_j^{(k)})_{j=1}^{\infty}\in\ell_q^u(E_k)$.

This class, denoted by $\mathcal{L}_{as(p,q)}^{n}$, forms a Banach multi-ideal but defects similar to those of $\mathcal{P}_{as(p,q)}^{n}$ can be easily found. So we easily have:

Example 6.1. *The sequence* $\left(\left(\mathcal{P}_{as,1}^{n},\|.\|_{as,1}\right),\left(\mathcal{L}_{as,1}^{n},\|.\|_{as,1}\right)\right)_{n=1}^{\infty}$ *is not coherent and not compatible with* Π_1.

The main problems of the classes above disappear when we slightly modify their definitions, as we see in the next definition:

Definition 6.2. *Let $1 \leq q \leq p < \infty$. An n-linear operator $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is everywhere absolutely $(p;q)$-summing (notation $\mathcal{L}^{n,ev}_{as(p;q)}(E_1, \ldots, E_n; F)$) if there exists $C \geq 0$ such that*

$$\left(\sum_{j=1}^{\infty} \left\| T(a_1 + x_j^{(1)}, \ldots, a_n + x_j^{(n)}) - T(a_1, \ldots, a_n) \right\|^p \right)^{1/p} \leq C \prod_{k=1}^{n} \left(\|a_k\| + \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,q} \right)$$

for all $(a_1, \ldots, a_n) \in E_1 \times \cdots \times E_n$ and $(x_j^{(k)})_{j=1}^{\infty} \in \ell_q^u(E_k)$, $k = 1, \ldots, n$. Moreover, the infimum of all C for which the inequality holds defines a complete norm on $\mathcal{L}^{n,ev}_{as(p;q)}$ denoted by $\|\cdot\|_{ev^{(2)}(p;q)}$.

The definition above is justified by the following result [3, Theorem 4.1]:

Theorem 6.3. *The following assertions are equivalent for $T \in \mathcal{L}(E_1, \ldots, E_n; F)$:*

(a) $T \in \mathcal{L}^{n,ev}_{as(p;q)}(E_1, \ldots, E_n; F)$.

(b) The sequence $\left(T(a_1 + x_j^{(1)}, ..., a_n + x_j^{(n)}) - T(a_1, ..., a_n) \right)_{j=1}^{\infty} \in \ell_p(F)$ for all $(x_j^{(k)})_{j=1}^{\infty} \in \ell_p^u(E_k)$ and all $(a_1, ..., a_n) \in E_1 \times \cdots \times E_n$.

For polynomials the definition and characterisation are similar (and equivalent, modulo norms), to the definition presented in Section 3:

Definition 6.4. *Let $1 \leq q \leq p < \infty$. A polynomial $P \in \mathcal{P}(^n E; F)$ is everywhere absolutely $(p;q)$-summing (notation $\mathcal{P}^{n,ev}_{as(p;q)}(^n E; F)$) if there exists $C \geq 0$ such that*

$$\left(\sum_{j=1}^{\infty} \|P(a + x_j) - P(a)\|^p \right)^{1/p} \leq C \left(\|a\| + \left\| (x_j)_{j=1}^{\infty} \right\|_{w,q} \right)^n$$

for all $a \in E$ and $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$. Moreover, the infimum of all C for which the inequality holds defines a complete norm on $\mathcal{P}^{n,ev}_{as(p;q)}(^n E; F)$ denoted by $\|.\|_{ev^{(2)}(p;q)}$.

As in the case of multilinear operators, the following characterisation holds [3, Theorem 4.2]:

Theorem 6.5. *The following assertions are equivalent for $P \in \mathcal{P}(^n E; F)$:*

(a) $P \in \mathcal{P}^{n,ev}_{as(p;q)}(^n E; F)$.

(b) The sequence $(P(a + x_j) - P(a))_{j=1}^{\infty} \in \ell_p(F)$ for all $(x_j)_{j=1}^{\infty} \in \ell_p^u(E)$ and all $a \in E$.

It was proved in [3, Proposition 4.3] that

$$\|A_n \colon \mathbb{K}^n \longrightarrow \mathbb{K} : A_n(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n\|_{ev^{(2)}(p;q)} = 1$$

for all $p \geq q \geq 1$. So, it is not difficult to show that $(\mathcal{L}^{n,ev}_{as(p;q)}, \|.\|_{ev^{(2)}(p;q)})$ is a Banach multi-ideal:

Proposition 6.6. $(\mathcal{L}^{n,ev}_{as(p;q)}, \|.\|_{ev^{(2)}(p;q)})_{n=1}^{\infty}$ *is a Banach multi-ideal.*

Proof. Let $u_j \in \mathcal{L}(G_j, E_j)$, $j = 1, \ldots, n$, $T \in \mathcal{L}^{n,ev}_{as(p;q)}(E_1, \ldots, E_n; F)$ and $w \in \mathcal{L}(F; G)$. Note that

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty}\left\|w\circ T\circ(u_1,\ldots,u_n)\left(a_1+x_j^{(1)},\ldots,a_n+x_j^{(n)}\right)-w\circ T\circ(u_1,\ldots,u_n)(a_1,\ldots,a_n)\right\|^p\right)^{1/p}\\
&\leq \|w\|\left(\sum_{j=1}^{\infty}\left\|T\left(u_1(a_1)+u_1(x_j^{(1)}),\ldots,u_n(a_n)+u_n(x_j^{(n)})\right)-T\left(u_1(a_1),\ldots,u_n(a_n)\right)\right\|^p\right)^{1/p}\\
&\leq \|w\|\,\|T\|_{ev^{(2)}(p;q)}\prod_{k=1}^{n}\left(\|u_k(a_k)\|+\left\|\left(u_k\left(x_j^{(k)}\right)\right)_{j=1}^{\infty}\right\|_{w,q}\right)\\
&\leq \|w\|\,\|T\|_{ev^{(2)}(p;q)}\,\|u_1\|\cdots\|u_n\|\prod_{k=1}^{n}\left(\|a_k\|+\left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q}\right)
\end{aligned}$$

and it follows that

$$\|w\circ T\circ(u_1,\ldots,u_n)\|_{ev^{(2)}(p;q)} \leq \|w\|\,\|T\|_{ev^{(2)}(p;q)}\,\|u_1\|\cdots\|u_n\|\,.$$

The other properties of multi-ideals are easily verified. □

In general, the ideal $(\mathcal{P}^{n,ev}_{as(p;q)}, \|.\|_{ev^{(2)}(p;q)})_{n=1}^{\infty}$ has indeed good properties (see [53] for details). It was also proved in [3, Proposition 4.4] that one can also show that

$$\|P_n : \mathbb{K} \to \mathbb{K}; P_n(\lambda) = \lambda^n\|_{ev^{(2)}(p;q)} = 1$$

for all $p \geq q \geq 1$ and it is also not difficult to show that $(\mathcal{P}^{n,ev}_{as(p;q)}, \|.\|_{ev^{(2)}(p;q)})_{n=1}^{\infty}$ is a Banach polynomial ideal. Besides, in [3, Proposition 4.9] it is also proved that $(\mathcal{P}^{n,ev}_{as(p;q)}, \|.\|_{ev^{(2)}(p;q)})_{n=1}^{\infty}$ is a global holomorphy type. The main result of this section, Theorem 6.12, shows that, contrary to what happens to the sequence $\left(\left(\mathcal{P}^{n}_{as(p,q)}, \|.\|_{as(p;q)}\right), \left(\mathcal{L}^{n}_{as(p,q)}, \|.\|_{as(p;q)}\right)\right)_{n=1}^{\infty}$, the sequence

$$\left((\mathcal{P}^{n,ev}_{as(p;q)}, \|.\|_{ev^{(2)}(p;q)}), (\mathcal{L}^{n,ev}_{as(p;q)}, \|.\|_{ev^{(2)}(p;q)})\right)_{n=1}^{\infty}$$

is coherent and compatible with $\Pi_{p,q}$.

The following result is important for our purposes (note that this is a variation of [3, Proposition 3.5]):

Proposition 6.7. *A polynomial P belongs to $\mathcal{P}^{n,ev}_{as(p;q)}(^nE; F)$ if and only if $\overset{\vee}{P}$ belongs to $\mathcal{L}^{n,ev}_{as(p;q)}(^nE; F)$.*

Proof. Let us prove the nontrivial implication. Let $b_k \in E$ with $k = 1, ..., n$. Using the Polarization Formula (see [30, 48]) we have

$$\begin{aligned}
&n!\,2^n \left[\overset{\vee}{P}\left(b_1 + x_j^{(1)}, \ldots, b_n + x_j^{(n)}\right) - \overset{\vee}{P}(b_1, \ldots, b_n)\right] \\
&= \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_n P\left(\varepsilon_1 (b_1 + x_j^{(1)}) + \cdots + \varepsilon_n (b_n + x_j^{(n)})\right) - \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_n P\left(\varepsilon_1 b_1 + \cdots + \varepsilon_n b_n\right) \\
&= \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_n \left[P\left((\varepsilon_1 b_1 + \cdots + \varepsilon_n b_n) + (\varepsilon_1 x_j^{(1)} + \cdots + \varepsilon_n x_j^{(n)})\right) - P\left(\varepsilon_1 b_1 + \cdots + \varepsilon_n b_n\right)\right]
\end{aligned}$$

and the result follows. □

Proposition 6.8. *If $P \in \mathcal{P}_{as(p;q)}^{n,ev}\left({}^n E; F\right)$ and $\gamma \in E^*$, then $\gamma P \in \mathcal{P}_{as(p;q)}^{n+1,ev}\left({}^{n+1} E; F\right)$ and*

$$\|\gamma P\|_{ev^{(2)}(p;q)} \leq \|\gamma\| \, \|P\|_{ev^{(2)}(p;q)} . \tag{6.3}$$

Proof. Let $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$. Note that

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty} \|\gamma(a + x_j) P(a + x_j) - \gamma(a) P(a)\|^p\right)^{1/p} \\
&\leq |\gamma(a)| \left(\sum_{j=1}^{\infty} \|P(a + x_j) - P(a)\|^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} \|\gamma(x_j) P(a + x_j)\|^p\right)^{1/p} \\
&\leq \|\gamma\| \, \|a\| \, \|P\|_{ev^{(2)}(p;q)} \left(\|a\| + \left\|(x_j)_{j=1}^{\infty}\right\|_{w,q}\right)^n + \|P\| \left(\sup_j \|a + x_j\|^n\right) \left(\sum_{j=1}^{\infty} |\gamma(x_j)|^p\right)^{1/p} .
\end{aligned}$$

Since $q \leq p$,

$$\|P\|_{ev^{(2)}(p;q)} \geq \|P\| \text{ and } \sup_j \|a + x_j\| \leq \left(\|a\| + \left\|(x_j)_{j=1}^{\infty}\right\|_{w,q}\right),$$

we have

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty} \|\gamma(a + x_j) P(a + x_j) - \gamma(a) P(a)\|^p\right)^{1/p} \\
&\leq \|\gamma\| \, \|a\| \, \|P\|_{ev^{(2)}(p;q)} \left(\|a\| + \left\|(x_j)_{j=1}^{\infty}\right\|_{w,q}\right)^n \\
&+ \|P\|_{ev^{(2)}(p;q)} \left(\|a\| + \left\|(x_j)_{j=1}^{\infty}\right\|_{w,q}\right)^n \|\gamma\| \left\|(x_j)_{j=1}^{\infty}\right\|_{w,q} \\
&= \|\gamma\| \, \|P\|_{ev^{(2)}(p;q)} \left(\|a\| + \left\|(x_j)_{j=1}^{\infty}\right\|_{w,q}\right)^{n+1}
\end{aligned}$$

and we get (6.3). □

Proposition 6.9. *If $P \in \mathcal{P}_{as(p;q)}^{n+1,ev}\left({}^{n+1}E;F\right)$ and $a \in E$, then $P_a \in \mathcal{P}_{as(p;q)}^{n,ev}\left({}^{n}E;F\right)$ and*

$$\left\|P_a\right\|_{ev^{(2)}(p;q)} \leq \left\|\overset{\vee}{P}\right\|_{ev^{(2)}(p;q)} \left\|a\right\|. \tag{6.4}$$

Proof. Let $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$ and $b \in E$. We just need to note that

$$\begin{aligned}
\left(\sum_{j=1}^{\infty}\left\|P_a(b+x_j)-P_a(b)\right\|^p\right)^{1/p} &= \left(\sum_{j=1}^{\infty}\left\|\overset{\vee}{P}(a,(b+x_j)^n)-\overset{\vee}{P}(a,b^n)\right\|^p\right)^{1/p} \\
&= \left(\sum_{j=1}^{\infty}\left\|\overset{\vee}{P}(a+0,(b+x_j)^n)-\overset{\vee}{P}(a,b^n)\right\|^p\right)^{1/p} \\
&\leq \left\|\overset{\vee}{P}\right\|_{ev^{(2)}(p;q)}\left(\left\|a\right\|+\left\|(0)_{j=1}^{\infty}\right\|_{w,q}\right)\left(\left\|b\right\|+\left\|(x_j)_{j=1}^{\infty}\right\|_{w,q}\right)^n.
\end{aligned}$$

□

Proposition 6.10. *Let $i \in \{1,\ldots,n+1\}$. If $T \in \mathcal{L}_{as(p;q)}^{n+1,ev}\left(E_1,\ldots,E_{n+1};F\right)$ and $a_i \in E_i$, then $T_{a_i} \in \mathcal{L}_{as(p;q)}^{ev}\left(E_1,\ldots,E_{i-1},E_{i+1},\ldots,E_{n+1};F\right)$ and*

$$\left\|T_{a_i}\right\|_{ev^{(2)}(p;q)} \leq \left\|T\right\|_{ev^{(2)}(p;q)}\left\|a_i\right\|. \tag{6.5}$$

Proof. Let $\left(x_j^{(k)}\right)_{j=1}^{\infty} \in l_q^u\left(E_k\right)$ and $a_j \in E_j$ for all $j \neq i$. The proof follows from the inequalities

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty}\left\|\begin{array}{c} T_{a_i}\left(a_1+x_j^{(1)},\ldots,a_{i-1}+x_j^{(i-1)},a_{i+1}+x_j^{(i+1)},\ldots,a_{n+1}+x_j^{(n+1)}\right) \\ -T_{a_i}\left(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{n+1}\right)\end{array}\right\|^p\right)^{1/p} \\
&= \left(\sum_{j=1}^{\infty}\left\|\begin{array}{c} T\left(a_1+x_j^{(1)},\ldots,a_{i-1}+x_j^{(i-1)},a_i+0,a_{i+1}+x_j^{(i+1)},\ldots,a_{n+1}+x_j^{(n+1)}\right) \\ -T\left(a_1,\ldots,a_{n+1}\right)\end{array}\right\|^p\right)^{1/p} \\
&\leq \left\|T\right\|_{ev^{(2)}(p;q)}\left\|a_i\right\|\prod_{\substack{k=1 \\ k\neq i}}^{n+1}\left(\left\|a_k\right\|+\left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q}\right).
\end{aligned}$$

□

Proposition 6.11. *If $T \in \mathcal{L}_{as(p;q)}^{n,ev}\left(E_1,\ldots,E_n;F\right)$ and $\gamma \in E_{k+1}^{*}$, then*

$$\gamma T \in \mathcal{L}_{as(p;q)}^{n+1,ev}\left(E_1,\ldots,E_{n+1};F\right) \text{ and } \left\|\gamma T\right\|_{ev^{(2)}(p;q)} \leq \left\|\gamma\right\|\left\|T\right\|_{ev^{(2)}(p;q)}. \tag{6.6}$$

Proof. Let $\left(x_j^{(k)}\right)_{j=1}^{\infty} \in l_q^u\left(E_k\right)$ and $\left(a_1, \ldots, a_{n+1}\right) \in E_1 \times \cdots \times E_{n+1}$. Then,

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty}\left\|\gamma\left(a_{n+1}+x_j^{(n+1)}\right) T\left(a_1+x_j^{(1)}, \ldots, a_n+x_j^{(n)}\right)-\gamma\left(a_{n+1}\right) T\left(a_1, \ldots, a_n\right)\right\|^p\right)^{1/p} \\
&\leq\left|\gamma\left(a_{n+1}\right)\right|\left(\sum_{j=1}^{\infty}\left\|T\left(a_1+x_j^{(1)}, \ldots, a_n+x_j^{(n)}\right)-T\left(a_1, \ldots, a_n\right)\right\|^p\right)^{1/p} \\
&+\left(\sum_{j=1}^{\infty}\left\|\gamma\left(x_j^{(n+1)}\right) T\left(a_1+x_j^{(1)}, \ldots, a_n+x_j^{(n)}\right)\right\|^p\right)^{1/p} \\
&\leq\left|\gamma\left(a_{n+1}\right)\right|\left(\sum_{j=1}^{\infty}\left\|T\left(a_1+x_j^{(1)}, \ldots, a_n+x_j^{(n)}\right)-T\left(a_1, \ldots, a_n\right)\right\|^p\right)^{1/p} \\
&+\|T\| \sup_j \prod_{k=1}^n\left(\left\|a_k+x_j^{(k)}\right\|\right)\left(\sum_{j=1}^{\infty}\left|\gamma\left(x_j^{(n+1)}\right)\right|^p\right)^{1/p} .
\end{aligned}$$

Using the same arguments of Proposition 6.8 we have

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty}\left\|\gamma\left(a_{n+1}+x_j^{(n+1)}\right) T\left(a_1+x_j^{(1)}, \ldots, a_n+x_j^{(n)}\right)-\gamma\left(a_{n+1}\right) T\left(a_1, \ldots, a_n\right)\right\|^p\right)^{1/p} \\
&\leq\|\gamma\|\left\|a_{n+1}\right\|\|T\|_{ev^{(2)}(p;q)} \prod_{k=1}^n\left(\left\|a_k\right\|+\left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q}\right) \\
&+\|T\|_{ev^{(2)}(p;q)} \prod_{k=1}^n\left(\left\|a_k\right\|+\left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q}\right)\|\gamma\|\left\|\left(x_j^{(n+1)}\right)_{j=1}^{\infty}\right\|_{w,q} \\
&=\|\gamma\|\|T\|_{ev^{(2)}(p;q)} \prod_{k=1}^{n+1}\left(\left\|a_k\right\|+\left\|\left(x_j^{(k)}\right)_{j=1}^{\infty}\right\|_{w,q}\right)
\end{aligned}$$

and the proof is done. □

The coherence and compatibility of $\left((\mathcal{P}_{as(p;q)}^{n,ev},\|.\|_{ev^{(2)}(p;q)}),(\mathcal{L}_{as(p;q)}^{n,ev},\|.\|_{ev^{(2)}(p;q)})\right)_{n=1}^{\infty}$ is a consequence of the previous results and Remark 3.3:

Theorem 6.12. *The sequence* $\left((\mathcal{P}_{as(p;q)}^{n,ev},\|.\|_{ev^{(2)}(p;q)}),(\mathcal{L}_{as(p;q)}^{n,ev},\|.\|_{ev^{(2)}(p;q)})\right)_{n=1}^{\infty}$ *is coherent and compatible with* $\Pi_{p,q}$.

7. Strongly summing multilinear operators and polynomials

The multi-ideal of strongly p-summing multilinear operators is one of the classes that best inherits the spirit of the ideal of absolutely p-summing linear operators (for papers comparing the different nonlinear extensions of absolutely summing operators we refer to [17, 53, 59]).

If $p \geq 1$, $T \in \mathcal{L}(E_1, ..., E_n; F)$ is strongly p-summing ($T \in \mathcal{L}\Pi_p^{n,\mathrm{str}}(E_1, ..., E_n; F)$) if there exists a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{m} \| T(x_j^{(1)}, ..., x_j^{(n)}) \|^p \right)^{1/p} \leq C \left(\sup_{\phi \in B_{\mathcal{L}(E_1,...,E_n;\mathbb{K})}} \sum_{j=1}^{m} | \phi(x_j^{(1)}, ..., x_j^{(n)}) |^p \right)^{1/p} . \tag{7.1}$$

for all $m \in \mathbb{N}$, $x_j^{(l)} \in E_l$ with $l = 1, ..., n$ and $j = 1, ..., m$. The infimum of all $C \geq 0$ satisfying (7.1) defines a complete norm, denoted by $\|.\|_{\mathcal{L}\Pi_{p,q}^{n,\mathrm{str}}}$, on the space $\mathcal{L}\Pi_{p,q}^{n,\mathrm{str}}(E_1, ..., E_n; F)$.

This concept is due to V. Dimant [29]. In the same paper the author proposes a definition for the polynomial case, but as mentioned in [19] this concept does not generate a polynomial ideal compatible with Π_p.

It is easy to prove that the ideal of strongly p-summing multilinear operators is closed under differentiation, closed for scalar multiplication. Besides, for this class we have a Grothendieck-type theorem and a Pietsch-Domination type theorem:

Theorem 7.1. *([29]) If $n \geq 2$, then*

$$\mathcal{L}(^n\ell_1; \ell_2) = \mathcal{L}\Pi_{1,1}^{n,str}(^n\ell_1; \ell_2).$$

Theorem 7.2. *([29]) $T \in \mathcal{L}(E_1, ..., E_n; F)$ is strongly p-summing if, and only if, there are a probability measure μ on $B_{(E_1 \otimes_\pi \cdots \otimes_\pi E_n)^*}$, with the weak-star topology, and a constant $C \geq 0$ so that*

$$\|T(x_1, ..., x_n)\| \leq C \left(\int_{B_{(E_1 \otimes_\pi \cdots \otimes_\pi E_n)^*}} |\varphi(x_1 \otimes \cdots \otimes x_n)|^p \, d\mu(\varphi) \right)^{1/p} \tag{7.2}$$

for all $(x_1, ..., x_n) \in E_1 \times \cdots \times E_n$.

As a consequence, there is an Inclusion Theorem (if $p \leq q$ then $\mathcal{L}\Pi_p^{n,\mathrm{str}} \subset \mathcal{L}\Pi_q^{n,\mathrm{str}}$). It is worth mentioning that even the fashionable multi-ideal of multiple summing multilinear operators (see [46]) does not have all these properties (for example, Pérez-García [58] proved that the inclusion theorem is not valid, in general). In the recent paper [53] a notion of "desired generalisation of absolutely summing operators" to the multilinear setting was discussed and the class of strongly p-summing multilinear operators seemed to be one of the "closest to perfection".

We will choose a concept of strongly p-summing polynomials different from the one from [19, 29]; we will consider that $P \in \mathcal{P}(^nE; F)$ is strongly p-summing (notation $\mathcal{P}\Pi_p^{n,\mathrm{str}}$) if $\overset{\vee}{P}$ belongs to $\mathcal{L}\Pi_p^{n,\mathrm{str}}(^nE; F)$ and

$$\|P\|_{\mathcal{P}\Pi_p^{n,\mathrm{str}}} := \left\| \overset{\vee}{P} \right\|_{\mathcal{L}\Pi_p^{n,\mathrm{str}}}.$$

So, $\left(\mathcal{P}\Pi_p^{n,\mathrm{str}}, \|.\|_{\mathcal{P}\Pi_p^{n,\mathrm{str}}}\right)_{n=1}^{\infty}$ is a Banach polynomial ideal and it is easy to prove that the sequence $\left(\mathcal{P}\Pi_p^{n,\mathrm{str}}, \|.\|_{\mathcal{P}\Pi_p^{n,\mathrm{str}}}\right)_{n=1}^{\infty}$ is a global holomorphy type. As we will see in the forthcoming Theorem 7.7, the sequence $\left(\left(\mathcal{P}\Pi_p^{n,\mathrm{str}}, \|.\|_{\mathcal{P}\Pi_p^{n,\mathrm{str}}}\right), \left(\mathcal{L}\Pi_p^{n,\mathrm{str}}, \|.\|_{\mathcal{L}\Pi_p^{n,\mathrm{str}}}\right)\right)_{n=1}^{\infty}$ is coherent and compatible with Π_p, and this result is quite adequate in view of the very good properties of the ideal of strongly summing multilinear mappings. For its proof we need to prove four simple propositions:

Proposition 7.3. *Let $k \in \{1, \ldots, n+1\}$. If $T \in \mathcal{L}\Pi_p^{n+1,str}(E_1, \ldots, E_{n+1}; F)$ and $a_k \in E_k$, then*

$$T_{a_k} \in \mathcal{L}\Pi_p^{n,str}(E_1, \ldots, E_n; F) \text{ and } \|T_{a_k}\|_{\mathcal{L}\Pi_p^{n,str}} \leq \|a_k\| \, \|T\|_{\mathcal{L}\Pi_p^{n+1,str}}.$$

Proof. The case $a_k = 0$ is immediate. Let us suppose $a_k \neq 0$. We just have to note that

$$\begin{aligned}
&\left(\sum_{j=1}^{m} \left\|T_{a_k}\left(x_j^{(1)}, \ldots, x_j^{(k-1)}, x_j^{(k+1)}, \ldots, x_j^{(n)}\right)\right\|^p\right)^{1/p} \\
&= \left(\sum_{j=1}^{m} \left\|T\left(x_j^{(1)}, \ldots, x_j^{(k-1)}, a_k, x_j^{(k+1)}, \ldots, x_j^{(n)}\right)\right\|^p\right)^{1/p} \\
&\leq \|T\|_{\mathcal{L}\Pi_p^{n+1,\mathrm{str}}} \|a_k\| \sup_{\varphi \in B_{\mathcal{L}(E_1,\ldots,E_n;\mathbb{K})}} \left(\sum_{j=1}^{m} \left|\varphi\left(x_j^{(1)}, \ldots, x_j^{(k-1)}, \frac{a_k}{\|a_k\|}, x_j^{(k+1)}, \ldots, x_j^{(n)}\right)\right|^p\right)^{1/p} \\
&\leq \|T\|_{\mathcal{L}\Pi_p^{n+1,\mathrm{str}}} \|a_k\| \sup_{\varphi \in B_{\mathcal{L}(E_1,\ldots,E_{k-1},E_{k+1},\ldots,E_{n+1};\mathbb{K})}} \left(\sum_{j=1}^{m} \left|\varphi\left(x_j^{(1)}, \ldots, x_j^{(k-1)}, x_j^{(k+1)}, \ldots, x_j^{(n)}\right)\right|^p\right)^{1/p}.
\end{aligned}$$

□

Proposition 7.4. *If $T \in \mathcal{L}\Pi_p^{n,str}(E_1, \ldots, E_n; F)$ and $\gamma \in E_{n+1}^*$, then*

$$\gamma T \in \mathcal{L}\Pi_p^{n+1,str}(E_1, \ldots, E_{n+1}; F) \text{ and } \|\gamma T\|_{\mathcal{L}\Pi_p^{n+1,str}} \leq \|\gamma\| \, \|T\|_{\mathcal{L}\Pi_p^{n,str}}.$$

Proof. Since $T \in \mathcal{L}\Pi_p^{n,\mathrm{str}}(E_1, \ldots, E_n; F)$, we have

$$\begin{aligned}
&\left(\sum_{j=1}^{m} \| \, T(x_j^{(1)}, ..., x_j^{(n)})\gamma\left(x_j^{(n+1)}\right) \|^p\right)^{1/p} \\
&\leq \|T\|_{\mathcal{L}\Pi_p^{n,\mathrm{str}}} \left(\sup_{\phi \in B_{\mathcal{L}(E_1,\ldots,E_n;\mathbb{K})}} \sum_{j=1}^{m} | \, \phi(x_j^{(1)}, ..., x_j^{(n)})\gamma\left(x_j^{(n+1)}\right) |^p\right)^{1/p} \\
&\leq \|\gamma\| \, \|T\|_{\mathcal{L}\Pi_p^{n,\mathrm{str}}} \left(\sup_{\psi \in B_{\mathcal{L}(E_1,\ldots,E_{n+1};\mathbb{K})}} \sum_{j=1}^{m} | \, \psi(x_j^{(1)}, ..., x_j^{(n+1)}) |^p\right)^{1/p}
\end{aligned}$$

and the proof is done. $\square$

Proposition 7.5. *If $P \in \mathcal{P}\Pi_p^{n+1,str}\left({}^{n+1}E;F\right)$ and $a \in E$, then P_a belongs to $\mathcal{P}\Pi_p^{n,str}\left({}^{n}E;F\right)$ and*

$$\|P_a\|_{\mathcal{P}\Pi_p^{n,str}} \leq \|a\| \left\|\overset{\vee}{P}\right\|_{\mathcal{L}\Pi_p^{n+1,str}}.$$

Proof. Since $\overset{\vee}{P} \in \mathcal{L}\Pi_p^{n+1,\mathrm{str}}\left({}^{n}E;F\right)$, from Proposition 7.3 we have

$$\overset{\vee}{P}_a \in \mathcal{L}\Pi_p^{n,\mathrm{str}}\left({}^{n}E;F\right)$$

and

$$\left\|\overset{\vee}{P}_a\right\|_{\mathcal{L}\Pi_p^{n,\mathrm{str}}} \leq \|a\| \left\|\overset{\vee}{P}\right\|_{\mathcal{L}\Pi_p^{n+1,\mathrm{str}}}.$$

Hence

$$\|P_a\|_{\mathcal{P}\Pi_p^{n,\mathrm{str}}} = \left\|(P_a)^{\vee}\right\|_{\mathcal{L}\Pi_p^{n,\mathrm{str}}} = \left\|\overset{\vee}{P}_a\right\|_{\mathcal{L}\Pi_p^{n,\mathrm{str}}} \leq \|a\| \left\|\overset{\vee}{P}\right\|_{\mathcal{L}\Pi_p^{n+1,\mathrm{str}}}.$$

$\square$

Proposition 7.6. *If $P \in \mathcal{P}\Pi_p^{n,str}\left({}^{n}E;F\right)$ and $\gamma \in E^*$, then γP belongs to $\mathcal{P}\Pi_p^{n+1,str}\left({}^{n+1}E;F\right)$ and*

$$\|\gamma P\|_{\mathcal{P}\Pi_p^{n+1,str}} \leq \|\gamma\| \left\|\overset{\vee}{P}\right\|_{\mathcal{L}\Pi_p^{n,str}}.$$

Proof. Since

$$(\gamma P)^{\vee}(x_1,...,x_{n+1}) = \frac{\gamma(x_1)\overset{\vee}{P}(x_2,...,x_{n+1}) + \cdots + \gamma(x_{n+1})\overset{\vee}{P}(x_1,...,x_n)}{n+1},$$

from Proposition 7.4 we have

$$\|\gamma P\|_{\mathcal{P}\Pi_p^{n+1,\mathrm{str}}} = \left\|(\gamma P)^{\vee}\right\|_{\mathcal{L}\Pi_p^{n+1,\mathrm{str}}} \leq \left\|\gamma\overset{\vee}{P}\right\|_{\mathcal{L}\Pi_p^{n+1,\mathrm{str}}} \leq \|\gamma\| \left\|\overset{\vee}{P}\right\|_{\mathcal{L}\Pi_p^{n,\mathrm{str}}}.$$

$\square$

As in the previous sections, the following theorem is a consequence of the previous propositions and Remark 3.3:

Theorem 7.7. *The sequence $\left(\left(\mathcal{P}\Pi_p^{n,str}, \|.\|_{\mathcal{P}\Pi_p^{n,str}}\right), \left(\mathcal{L}\Pi_p^{n,str}, \|.\|_{\mathcal{L}\Pi_p^{n,str}}\right)\right)_{n=1}^{\infty}$ is coherent and compatible with Π_p.*

8. Multiple summing polynomials and multilinear operators

If $1 \leq q \leq p < \infty$ and n is a positive integer, an n-linear operator $T : E_1 \times \cdots \times E_n \to F$ is multiple $(p;q)$-summing ($T \in \mathcal{L}\Pi_{p,q}^{n,\text{mult}}(E_1, ..., E_n; F)$) if there exists $C > 0$ such that

$$\left(\sum_{j_1,...,j_n=1}^{\infty} \|T(x_{j_1}^{(1)}, ..., x_{j_n}^{(n)})\|^p \right)^{1/p} \leq C \prod_{k=1}^{n} \|(x_j^{(k)})_{j=1}^{\infty}\|_{w,q} \tag{8.1}$$

for all $(x_j^{(k)})_{j=1}^{\infty} \in \ell_q^w(E_k)$, $k = 1, ..., n$. The infimum of all $C \geq 0$ satisfying (8.1) defines a complete norm, denoted by $\|.\|_{\mathcal{L}\Pi_{p,q}^{n,\text{mult}}}$, on the space $\mathcal{L}\Pi_{p,q}^{n,\text{mult}}(E_1, ..., E_n; F)$.

This class was introduced by M.C. Matos [46] and, independently, by F. Bombal, D. Pérez-García and I. Villanueva [7] and was investigated by different authors in recent years (see, for example [12, 24, 58, 65]).

The ideal of multiple summing polynomials is defined as in Definition 4.1 and denoted by $\mathcal{P}\Pi_{p,q}^{n,\text{mult}}$ (and the norm is denoted by $\|.\|_{\mathcal{P}\Pi_{p,q}^{n,\text{mult}}}$). By using essentially the same arguments from the previous section we can prove:

Theorem 8.1. *The sequence* $\left(\left(\mathcal{P}\Pi_{p,q}^{n,mult}, \|.\|_{\mathcal{P}\Pi_{p,q}^{n,mult}} \right), \left(\mathcal{L}\Pi_{p,q}^{n,mult}, \|.\|_{\mathcal{L}\Pi_{p,q}^{n,mult}} \right) \right)_{n=1}^{\infty}$ *is coherent and compatible with* $\Pi_{p,q}$.

We finish this short section with an illustrative example on how the concepts of coherence and compatibility work well together.

Example 8.2. *For all* n *let* $\mathcal{U}_1 = \mathcal{M}_1 = \Pi_{1,1}$ *and consider the artificial sequence* $(\mathcal{U}_n, \mathcal{M}_n)_{n=1}^{\infty}$ *defined by*

$$\begin{aligned}
(\mathcal{U}_{2n}, \mathcal{M}_{2n})_{n=1}^{\infty} &= \left(\left(\mathcal{P}\Pi_{1,1}^{2n,mult}, \|.\|_{\mathcal{P}\Pi_{1,1}^{2n,mult}} \right), \left(\mathcal{L}\Pi_{1,1}^{2n,mult}, \|.\|_{\mathcal{L}\Pi_{1,1}^{2,mult}} \right) \right)_{n=1}^{\infty} \\
(\mathcal{U}_{2n+1}, \mathcal{M}_{2n+1})_{n=1}^{\infty} &= \left(\left(\mathcal{P}\Pi_{1,1}^{2n+1,str}, \|.\|_{\mathcal{P}\Pi_{1,1}^{2n+1,str}} \right), \left(\mathcal{L}\Pi_{1,1}^{2n+1,str}, \|.\|_{\mathcal{L}\Pi_{1,1}^{2n+1,str}} \right) \right)_{n=1}^{\infty}.
\end{aligned}$$

From our previous results it is easy to see that the sequence $(\mathcal{U}_n, \mathcal{M}_n)_{n=1}^{\infty}$ *is compatible with* $\Pi_{1,1}$. *On the other hand it is also not difficult to show that* $(\mathcal{U}_n, \mathcal{M}_n)_{n=1}^{\infty}$ *is not coherent. So the concepts of coherent and compatible pairs, together, besides filtering sequences that keep the spirit of a given operator ideal, also seem to be an adequate method of avoiding artificial constructions.*

9. Strongly coherent and compatible sequences

An apparently stronger notion of coherence and compatibility can be considered if we replace (CP2) and (CH2) by (respectively)

(CP2*) There is a constant $\alpha_2 > 0$ so that if $P \in \mathcal{U}_n({}^nE; F)$ and $a \in E$, then $P_{a^{n-1}} \in \mathcal{U}(E; F)$ and

$$\|P_{a^{n-1}}\|_{\mathcal{U}} \leq \alpha_2 \|P\|_{\mathcal{U}_n} \|a\|^{n-1}.$$

(CH2*) There is a constant $\beta_2 > 0$ so that if $P \in \mathcal{U}_{k+1}\left({}^{k+1}E;F\right)$, $a \in E$, then P_a belongs to $\mathcal{U}_k\left({}^{k}E;F\right)$ and

$$\|P_a\|_{\mathcal{U}_k} \leq \beta_2 \|P\|_{\mathcal{U}_{k+1}} \|a\| .$$

This approach is closer to the original concepts from [19] for polynomial ideals. For the cases investigated in the Sections 4, 7 and 8 there is no difference between the concepts since

$$\|P\|_{\mathcal{U}_n} = \left\|\overset{\vee}{P}\right\|_{\mathcal{M}_n} .$$

The case of the "canonical" pairs $(\mathcal{P}_k, \mathcal{L}_k)_{k=1}^{\infty}$ (composed by the ideals of continuous n-homogeneous polynomials and continuous n-linear operators, with the sup norm) illustrates that the notion of strongly coherent pairs has important restrictions. In fact, from [10, Proposition 8.5] we know that (CH2*) is not valid for this case when dealing with real scalars.

However, for the case of r-dominated multilinear operators and polynomials we remark that strong coherence and compatibility holds.

From now on $n \in \mathbb{N}$ and $r \in [n, \infty]$.

Definition 9.1. *A multilinear operator $T : E_1 \times \cdots \times E_n \to F$ is r-dominated (in this case we write $T \in \mathcal{L}_{d,r}^n (E_1, ..., E_n; F)$) if there exists a constant $C > 0$ such that*

$$\left(\sum_{j=1}^{m} \left\|T\left(x_j^{(1)}, \ldots, x_j^{(n)}\right)\right\|^{r/n}\right)^{n/r} \leq C \prod_{j=1}^{n} \left\|\left(x_i^{(j)}\right)_{i=1}^{m}\right\|_{w,r}$$

for all $m \in \mathbb{N}$ and $x_1^{(j)}, \ldots, x_m^{(j)} \in E_j$. The smallest such C is denoted by $\|T\|_{d,r}$. It is well-known that $\left(\mathcal{L}_{d,r}, \|\cdot\|_{d,r}\right)$ is a Banach ideal of multilinear mappings (recall that $n \leq r$).

The terminology "dominated" is motivated by the Pietsch Domination Theorem:

Theorem 9.2 (Pietsch, Geiss, 1985)**.** *([33]) $T \in \mathcal{L}(E_1, ..., E_n; F)$ is r-dominated if and only if there exist $C \geq 0$ and probability measures μ_j on the Borel σ-algebras of $B_{E_j^*}$ endowed with the weak star topologies such that*

$$\|T(x_1, ..., x_n)\| \leq C \prod_{j=1}^{n} \left(\int_{B_{E_j^*}} |\varphi(x_j)|^p \, d\mu_j(\varphi)\right)^{1/p}$$

for all $x_j \in E_j$ and $j = 1, ..., n$. Moreover, the infimum of the C that satisfy the inequality above is precisely $\|T\|_{d,r}$.

For recent generalisations of the Pietsch Domination Theorem and related results we mention [54, 55, 57]. The concept of r-dominated polynomial is similar, *mutatis mutandis*, to the notion for multilinear operators:

Definition 9.3. *An n-homogeneous polynomial $P : E \to F$ is r-dominated (in this case we write $P \in \mathcal{P}^n_{d,r}({}^nE;F)$), if there is a constant $C > 0$ such that*

$$\left(\sum_{i=1}^{m} \|P(x_i)\|^{r/n}\right)^{n/r} \leq C \, \|(x_i)_{i=1}^m\|_{w,r}^k$$

for all $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in E$. The smallest such C is denoted by $\|P\|_{d,r}$. It is well-known that $\left(\mathcal{P}_{d,r}, \|\cdot\|_{d,r}\right)$ is a Banach ideal of polynomials.

The next result is folklore:

Proposition 9.4. *A polynomial P belongs to $\mathcal{P}^n_{d,r}({}^nE;F)$ if and only if $\overset{\vee}{P}$ belongs to $\mathcal{L}^n_{d,r}({}^nE;F)$.*

From [19] we know that (CP2*), (CH2*), (CP4) and (CH4) are valid for $\left(\mathcal{P}^n_{d,r}, \|\cdot\|_{d,r}\right)_{n=1}^{\infty}$.

The other properties are easily verified. For example, let us check (CH3):

Proposition 9.5. *If $T \in \mathcal{L}^k_{d,r}(E_1, \ldots, E_k; F)$ and $\gamma \in E^*_{k+1}$, $\gamma T \in \mathcal{L}^{k+1}_{d,r}(E_1, \ldots, E_{k+1}; F)$ then*

$$\|\gamma T\|_{d,r} \leq \|\gamma\| \, \|T\|_{d,r} \, .$$

Proof. From the Pietsch Domination Theorem there are Borel probability measures $\mu_1, ..., \mu_k$ so that

$$\|T(x_1, ..., x_k)\| \leq \|T\|_{d,r} \prod_{i=1}^{k} \left(\int_{B_{E_i^*}} |\varphi_i(x_i)|^r \, d\mu_i\right)^{1/r}.$$

By considering the Dirac measure $\delta_{\gamma/\|\gamma\|}$ we have

$$\begin{aligned}
&\|\gamma(x_{k+1})T(x_1, ..., x_k)\| \\
&\leq \|\gamma\| \, \|T\|_{d,r} \left|\frac{\gamma}{\|\gamma\|}(x_{k+1})\right| \prod_{i=1}^{k} \left(\int_{B_{E_i^*}} |\varphi_i(x_i)|^r \, d\mu_i\right)^{1/r} \\
&= \|\gamma\| \, \|T\|_{d,r} \left(\int_{B_{E^*_{k+1}}} |\varphi_{k+1}(x_{k+1})|^r \, d\delta_{\gamma/\|\gamma\|}\right)^{1/r} \prod_{i=1}^{k} \left(\int_{B_{E_i^*}} |\varphi_i(x_i)|^r \, d\mu_i\right)^{1/r}
\end{aligned}$$

and again the Pietsch Domination Theorem asserts that

$$\|\gamma T\|_{d,r} \leq \|\gamma\| \, \|T\|_{d,r} \, .$$

□

So, we have:

Proposition 9.6. *The sequence $\left(\left(\mathcal{P}^n_{d,r}, \|.\|_{d,r}\right), \left(\mathcal{L}^n_{d,r}, \|.\|_{d,r}\right)\right)_{n=1}^{\infty}$ is strongly coherent and strongly compatible with Π_r.*

Remark 9.7. *Since $\|P\|_{d,r} \leq \left\|\overset{\vee}{P}\right\|_{d,r}$ for all $P \in \mathcal{P}^n_{d,r}$ it is obvious that the sequence*

$$\left(\left(\mathcal{P}^n_{d,r}, \|.\|_{d,r}\right), \left(\mathcal{L}^n_{d,r}, \|.\|_{d,r}\right)\right)_{n=1}^{\infty}$$

is also coherent and compatible with Π_r.

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Departamento de Matemática,
Universidade Federal da Paraíba,
58.051-900 - João Pessoa, Brazil.
E-mail address: dmpellegrino@gmail.com and pellegrino@pq.cnpq.br

Instituto de Matemática,
Universidade Federal da Bahia,
Av. Adhemar de Barros, s/n, Sala 268,
Salvador, 40170110, Brazil.
E-mail address: joilsonribeiro@yahoo.com.br